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Explosive resonant triads in a continuously stratified shear flow

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A variational formulation for three-dimensional waves in a continuously stratified shear flow is used to derive the equations governing a resonant triad of waves. It is argued that in general, critical layers are necessary for the existence of explosive resonant triads.

1. Introduction

In recent years, the interaction of positive and negative (disturbance) energy waves has been exploited to explain linear and nonlinear instability mechanisms in stratified shear flows (Cairns 1979; Craik & Adam 1979; Tsutahara 1984; Tsutahara & Hashimoto 1986; Ostrovskii, Rybak & Tsmring 1986; and Romanova & Shrira 1988). In particular, these authors have shown that explosive resonant triad interactions (i.e. triads of waves in which the wave of the largest frequency has oppositely signed energy to the other two triad members, whence the wave amplitudes develop a singularity in a finite time, cf. Craik 1985, §15) occur among interfacial waves in layered fluids with piecewise-constant or -linear velocity profiles and piecewise-constant or -exponential density profiles. Moreover, these explosive instabilities have been shown to occur in parametric domains in which the underlying flow can be linearly stable but in which any critical-layer singularities (i.e. the singularity that occurs when the phase speed of the wave equals the background flow speed) have been suppressed by virtue of the layered-fluid model.

As reported by Maslowe (1985) and Morland, Saffman & Yuen (1991), the stability characteristics of broken-line profiles can differ significantly from those of smooth profiles; hence, we consider here resonant triad interactions on smooth basic flow profiles. This study also presents a variational formulation suitable for future studies of the evolution of nonlinear oblique waves propagating through general curved velocity and density profiles. We begin, in §2, by developing this variational formulation for oblique waves propagating on a horizontal shear flow $U_0(z)$ with background density $\rho_0(z)$, where z is the coordinate parallel to gravity, in terms of Lagrangian coordinates advected with this basic flow. In §3, we derive the average Lagrangian for a single wave mode and show that a necessary condition for the existence of negative-energy waves is

$$\omega(\omega-\boldsymbol{\kappa}\cdot U_0(z)) < 0 \tag{1.1}$$

for some z in the flow, where ω is the wave frequency and $\boldsymbol{\kappa}$ is the horizontal wavenumber vector. We emphasize that while the sign of the energy of a single mode depends upon the reference frame from which it is viewed, the conditions under which an explosive resonant triad exist are Galilean invariant (Davidson 1972). In §4, we derive the equations for a resonant triad of waves in a general stratified shear flow and argue that at least one triad member must possess a critical layer for an explosive instability to occur. As we are interested in background flows that are linearly stable, it is likely that inviscid eigensolutions with nonlinear critical layers, which Maslowe (1973) has shown to exist when Ri, the local Richardson number, is everywhere greater than $\frac{1}{4}$ (a necessary condition for linear stability), will be relevant to this instability. Thus a numerical and/or analytical study of explosive resonant triads for continuous shear flows is in general complicated by the presence of the critical layer, and is under consideration. In Appendix A, we show how an interface across which there may be a density and velocity discontinuity may be incorporated into the present formulation. Hence, our formulation may be applied to flow profiles with general stratification and shear between interfaces as well as to continuous profiles.

In a recent article, Zhang (1991) has shown by direct computation for a particular Lagrangian and the fourth-order average Lagrangian is independent of the third-order trial function. In Appendix B, we present a general proof that the error in the average Lagrangian is the order of the square of the error of the trial function, which is relevant to our calculation in §4.

2. Lagrangian formulation

The motion of an inviscid, incompressible, stably stratified fluid is governed by

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\beta}\nabla' p - \frac{\rho}{\beta}\hat{\boldsymbol{k}}, \tag{2.1}$$

$$\nabla'\cdot\boldsymbol{u} = 0, \quad \frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \tag{2.2a, b}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla'. \tag{2.3}$$

Here $\boldsymbol{x}' = (x', y', z')$ are the Eulerian Cartesian coordinates (z' is the vertical coordinate and $\hat{\boldsymbol{k}}$ is a unit vector in the vertical direction), $\nabla' = (\partial/\partial x', \partial/\partial y', \partial/\partial z')$, $\boldsymbol{u} = (u, v, w)$ are the velocity components, p is the pressure and t is the time. The non-dimensional variables in (2.1)–(2.3) are related to the dimensional variables (indicated by hats) according to

$$\hat{t} = N_1^{-1} t, \quad \hat{\boldsymbol{x}}' = L\boldsymbol{x}', \quad \hat{\boldsymbol{u}} = N_1 L\boldsymbol{u}, \quad \hat{\rho} = \rho_1 \rho, \quad \hat{p} = \rho_1 g L p, \tag{2.4a–e}$$

where N_1 is a typical value of the Brunt–Väisälä frequency, L is a typical lengthscale (either wavelength or an appropriate vertical lengthscale), ρ_1 is a typical value of the density, and the parameter $\beta \equiv N_1^2 L/g$ in (2.1) is small in the Boussinesq approximation.

We consider a basic state consisting of a horizontal shear flow $\boldsymbol{U}_0 = (U_0(z), V_0(z), 0)$ and background density field $\rho_0(z)$ bounded by rigid planes at $z = \pm d$ and define particle displacements $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ so that the Eulerian coordinates $\boldsymbol{x}'$ are related to the Lagrangian coordinates $\boldsymbol{x}$ advected with this basic flow according to

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t). \tag{2.5}$$

Since the density is a material property in a non-diffusive, incompressible fluid, the Jacobian of the transformation from $\boldsymbol{x}$ to $\boldsymbol{x}'$ is equal to 1:

$$J \equiv \det\left(\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right) = 1, \tag{2.6a}$$

which implies that $$\nabla\cdot\xi+\tfrac{1}{2}\nabla\cdot\{\xi(\nabla\cdot\xi)-(\xi\cdot\nabla)\xi\}+\det\left(\frac{\partial\xi}{\partial x}\right)=0, \tag{2.6b}$$

where ∇ is the gradient operator in terms of the Lagrangian variables. The total time derivative (2.3) is given by

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+U_0\cdot\nabla, \tag{2.7}$$

whence (2.1) is transformed to

$$\rho_0\frac{\mathrm{D}^2\xi_i}{\mathrm{D}t^2}+\frac{1}{\beta}\frac{\partial}{\partial x_j}(K_{ij}p)+\frac{\rho_0}{\beta}\delta_{i3}=0, \tag{2.8}$$

where latin indices take the values 1, 2, or 3, repeated indices are summed over and K_{ij} is the i,jth co-factor of J that satisfies

$$K_{ij}\frac{\partial x_i'}{\partial x_k}=\delta_{jk}J=K_{ji}\frac{\partial x_k'}{\partial x_i}, \tag{2.9a}$$

or $$K_{ij}=\delta_{ij}\left\{1+\frac{\partial\xi_k}{\partial x_k}\right\}-\frac{\partial\xi_j}{\partial x_i}+\tfrac{1}{2}\epsilon_{ilm}\,\epsilon_{jpq}\frac{\partial\xi_l}{\partial x_p}\frac{\partial\xi_m}{\partial x_q}. \tag{2.9b}$$

In (2.9), δ_{ij} is the Kronecker delta function and ϵ_{ijk} is the usual permutation symbol. It also is useful to note that $\partial K_{ij}/\partial x_j=0$. We remark that the advective nonlinearity of (2.1) now appears in the pressure term of (2.8).

We next define a pressure perturbation $q(x,t)$ according to

$$p\equiv p_0(z')+\beta q=p_0(z+\zeta)+\beta q,\quad \frac{\mathrm{d}p_0(z)}{\mathrm{d}z}=-\rho_0(z). \tag{2.10a, b}$$

Substituting (2.10) into (2.8), we obtain

$$\rho_0\frac{\mathrm{D}^2\xi_i}{\mathrm{D}t^2}+\frac{\partial}{\partial x_j}(K_{ij}q)+\frac{1}{\beta}\{\rho_0(z)-\rho_0(z+\zeta)\}\delta_{i3}=0. \tag{2.11}$$

Equations (2.6*a*) and (2.11) provide four equations in the four unknowns ξ and q. These equations may be deduced from the variational principle

$$\delta A=0,\quad A\equiv\iiint L\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \tag{2.12}$$

where A is the action integral of the Lagrangian,

$$L=\int_{-d}^{d}\frac{1}{2}\rho_0\left|\frac{\mathrm{D}\xi_i}{\mathrm{D}t}\right|^2+q(J-1)-\frac{1}{\beta}\{\rho_0(z)\,\zeta+p_0(z+\zeta)-p_0(z)\}\,\mathrm{d}z. \tag{2.13}$$

Independent variations δq give (2.6*a*) and $\delta\xi$ give (2.11) and the boundary conditions

$$\zeta=0\quad(z=\pm d). \tag{2.14}$$

3. Single-mode average Lagrangian

We compute the average Lagrangian for a single modulated wave superimposed on the basic state described in §2. To this end, we pose an expansion of the form

$$\zeta=\alpha\,\mathrm{Re}\,[A(\chi,\tau)\,\phi(z)\,\mathrm{e}^{\mathrm{i}\theta}]+O(\alpha^2,\alpha\epsilon), \tag{3.1a}$$

$$(\xi,\eta,q)=\alpha\,\mathrm{Re}\left[(\mathrm{i}k,\mathrm{i}l,\rho_0\,\hat{\omega}^2)\frac{A(\chi,\tau)}{\kappa^2}\frac{\mathrm{d}\phi}{\mathrm{d}z}\mathrm{e}^{\mathrm{i}\theta}\right]+O(\alpha^2,\alpha\epsilon), \tag{3.1b}$$

where A is a slowly varying complex envelope,

$$\chi \equiv \epsilon(x,y), \quad \tau \equiv \epsilon t, \tag{3.2a, b}$$

are slow horizontal space and time scales,

$$\theta \equiv \epsilon^{-1}(\boldsymbol{\kappa}\cdot\boldsymbol{\chi}-\omega\tau), \quad \boldsymbol{\kappa} \equiv (k,l), \quad \kappa \equiv |\boldsymbol{\kappa}|, \tag{3.3a–c}$$

α and ϵ are small parameters, and $\phi(z)$ is the vertical modal structure of the wave and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\rho_0\,\hat{\omega}^2\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)+\rho_0(N^2-\hat{\omega}^2)\,\kappa^2\phi = 0 \tag{3.4}$$

subject to $$\phi = 0 \quad (z = \pm d), \tag{3.5}$$

where $$\hat{\omega}(z) \equiv \omega - \boldsymbol{\kappa}\cdot \boldsymbol{U}_0(z) \tag{3.6a}$$

is the intrinsic frequency and

$$N^2(z) = -\frac{\mathrm{d}\rho_0/\mathrm{d}z}{\beta\rho_0} \tag{3.6b}$$

is the non-dimensional Brunt–Väisälä frequency. We remark that (3.4) may be transformed to the Taylor–Goldstein equation (with non-Boussinesq terms) by invoking $\phi(z) = \mathrm{i}\Phi(z)/\hat{\omega}(z)$ where the vertical velocity is given by $w = \mathrm{Re}\,[A\Phi(z)\,\mathrm{e}^{\mathrm{i}\theta}]$. As we are interested in examining flows that are linearly stable, we henceforth assume that $\hat{\omega}$ and ϕ are real-valued.

Substituting (3.1) into (2.13) and averaging over a 2π interval of θ (this average is denoted by $\langle\ \rangle$), we obtain

$$\langle L\rangle = \alpha^2\mathscr{L}_0+\alpha^2\epsilon\mathscr{L}_1+\alpha^3\mathscr{L}_2+O(\alpha^4,\alpha^3\epsilon,\alpha^2\epsilon^2), \tag{3.7}$$

where $$\mathscr{L}_0 = D(\omega,\boldsymbol{\kappa})\,a^2, \quad A \equiv a\,\mathrm{e}^{\mathrm{i}\psi} \tag{3.8a, b}$$

and $$D(\omega,\boldsymbol{\kappa}) = \frac{1}{4\kappa^2}\int_{-d}^{d}\rho_0[\hat{\omega}^2\phi'^2+\kappa^2(\hat{\omega}^2-N^2)\,\phi^2]\,\mathrm{d}z. \tag{3.8c}$$

At $O(\alpha^2)$, independent variations of $\langle L\rangle$ with respect to a yield the dispersion relationship $D(\omega,\boldsymbol{\kappa}) = 0$. (It is straightforward to show directly from (3.4) that $D = 0$.) In Appendix A, we show how an interface across which there may be a density or velocity discontinuity may be incorporated into (3.8c).

Proceeding to $O(\alpha^2\epsilon)$, we find

$$\mathscr{L}_1 = -\tfrac{1}{2}\mathrm{i}[D_\omega(A\bar{A}_\tau-\bar{A}A_\tau)-D_{\boldsymbol{\kappa}}\cdot(A\bar{A}_\chi-\bar{A}A_\chi)] = (-D_\omega\psi_\tau+D_{\boldsymbol{\kappa}}\cdot\psi_\chi)\,a^2, \tag{3.9}$$

where the subscripts τ, $\boldsymbol{\kappa}$ and $\boldsymbol{\chi}$ indicate partial differentiation and the overbar indicates complex conjugation. Independent variations with respect to a and ψ yield the familiar results that in the linear approximation the envelope of the wave moves with the group velocity and that the wave action is conserved, respectively. In the present, single-mode approximation, $\mathscr{L}_2 = 0$.

The pseudoenergy of the wave, which is a measure of the energy of the disturbance, is given by (cf. Grimshaw 1984)

$$E \equiv \left\langle\frac{\partial\xi}{\partial t}\cdot\frac{\partial L}{\partial(\partial\xi/\partial t)}-L\right\rangle. \tag{3.10a}$$

Substituting (2.13) and (3.1) into (3.10a), we find

$$E = \alpha^2\omega D_\omega a^2+O(\alpha^3,\alpha\epsilon^2), \tag{3.10b}$$

where $$\omega D_{\omega} = \frac{1}{2}\frac{\omega}{\kappa^2}\int_{-d}^{d} \rho_0 \hat{\omega}(\phi'^2+\kappa^2\phi^2)\,\mathrm{d}z; \tag{3.10c}$$

hence, a necessary condition for the pseudoenergy to be negative is that $\omega\hat{\omega} < 0$ (i.e. (1.1)) somewhere in the flow.

4. Triad equations

We next consider the interaction of a triad of waves that satisfy the resonance conditions
$$\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3 = 0, \quad \omega_1+\omega_2+\omega_3 = 0, \tag{4.1a, b}$$
and choose an expansion of the form
$$\zeta = \alpha\,\mathrm{Re}\left[\sum_{r=1}^{3} A^{(r)}(\chi,\tau)\,\phi_r(z)\,\mathrm{e}^{\mathrm{i}\theta_r}\right]+O(\alpha^2,\alpha\epsilon), \tag{4.2a}$$
$$(\xi,\eta,q) = \alpha\,\mathrm{Re}\left[\sum_{r=1}^{3}(\mathrm{i}k_r,\mathrm{i}l_r,\rho_0\hat{\omega}_r^2)\frac{A^{(r)}(\chi,\tau)}{\kappa_r^2}\frac{\mathrm{d}\phi_r}{\mathrm{d}z}\mathrm{e}^{\mathrm{i}\theta_r}\right]+O(\alpha^2,\alpha\epsilon), \tag{4.2b}$$
where $$\hat{\omega}_r \equiv \omega_r-\boldsymbol{\kappa}_r\cdot\boldsymbol{U}_0, \tag{4.2c}$$
$A^{(r)}$ is the slowly varying complex envelope of the rth wave and the subscript r designates the triad member. We emphasize that only the leading-order approximations to ξ and q are necessary in the derivation of the triad equations even though at $O(\alpha^2)$, ξ and q have terms proportional to $\mathrm{e}^{\mathrm{i}(\theta_m+\theta_n)}$ $(n,m = 1,2,3, m \neq n)$ which through the interaction with the $O(\alpha)$ approximation to ξ and q proportional to $\mathrm{e}^{\mathrm{i}\theta_r}(r \neq m \neq n)$ appear to give an $O(\alpha^3)$ contribution to $\langle L\rangle$. That the third-order average Lagrangian is independent of the second-order trial functions for resonant triad interactions among capillary–gravity waves was shown by direct computation by Simmons (1969). Here, in Appendix B, we present a general proof showing that the error in the average Lagrangian is of the order of the square of the error in the trial function. While this proof follows from first principles and is not surprising (indeed, it has been implicitly invoked by many authors), it appears worthwhile to present since it does not appear to be universally appreciated. For example, Zhang (1991) has shown by direct computation that the fourth-order Lagrangian for short surface waves riding on long waves is independent of the third-order trial function.

We next substitute (4.2) into (2.13) and average over θ to obtain
$$\langle L\rangle = \alpha^2\sum_{r=1}^{3}\mathscr{L}_0^{(r)}+\alpha^2\epsilon\sum_{r=1}^{3}\mathscr{L}_1^{(r)}+\alpha^3\gamma(A^{(1)}A^{(2)}A^{(3)}+\bar{A}^{(1)}\bar{A}^{(2)}\bar{A}^{(3)})+O(\alpha^4,\alpha^3\epsilon,\alpha^2\epsilon^2), \tag{4.3}$$
where $\mathscr{L}_0^{(r)}$ and $\mathscr{L}_1^{(r)}$ are given by (3.8*a*) and (3.9) for the rth wave and
$$\begin{aligned}\gamma = \int_{-d}^{d}\frac{\rho_0}{8}\Bigg\{&\Bigg[-2\sum_{r=1}^{3}\hat{\omega}_r^2/\kappa_r^2+[\hat{\omega}_1^2(k_2 l_3-k_3 l_2)^2+\hat{\omega}_2^2(k_1 l_3-k_3 l_1)^2\\ &+\hat{\omega}_3^2(k_1 l_2-k_2 l_1)^2]/\kappa_1^2\kappa_2^2\kappa_3^2\Bigg]\phi_1'\phi_2'\phi_3'\\ &+\frac{\hat{\omega}_1^2}{\kappa_1^2}(\boldsymbol{\kappa}_2\cdot\boldsymbol{\kappa}_3)\,\phi_1'\left[\frac{\phi_2\phi_3''}{\kappa_3^2}+\frac{\phi_2''\phi_3}{\kappa_2^2}\right]+\frac{\hat{\omega}_2^2}{\kappa_2^2}(\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_3)\,\phi_2'\left[\frac{\phi_1\phi_3''}{\kappa_3^2}+\frac{\phi_1''\phi_3}{\kappa_1^2}\right]\\ &+\frac{\hat{\omega}_3^2}{\kappa_3^2}(\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_2)\,\phi_3'\left[\frac{\phi_1\phi_2''}{\kappa_2^2}+\frac{\phi_1''\phi_2}{\kappa_1^2}\right]+N^2(\phi_1\phi_2\phi_3)'\Bigg\}\,\mathrm{d}z.\end{aligned} \tag{4.4}$$

A method in which an interface may be incorporated into (4.4) is presented in Appendix A.

We now assume $\epsilon = \alpha$ to balance nonlinearity and modulation and invoke Hamilton's principle for the Lagrangian (4.3) to obtain at $O(\alpha^2)$ that each wave satisfies the dispersion relationship

$$D(\omega_r, \boldsymbol{\kappa}_r) = 0 \quad (r = 1, 2, 3), \tag{4.5}$$

and to obtain at $O(\alpha^3)$ the triad equations

$$A^{(1)}_\tau + V_1 \cdot A^{(1)}_\chi = \frac{\mathrm{i}\gamma}{D_\omega(\omega_1, \boldsymbol{\kappa}_1)} \bar{A}^{(2)} \bar{A}^{(3)}, \tag{4.6a}$$

$$A^{(2)}_\tau + V_2 \cdot A^{(2)}_\chi = \frac{\mathrm{i}\gamma}{D_\omega(\omega_2, \boldsymbol{\kappa}_2)} \bar{A}^{(1)} \bar{A}^{(3)}, \tag{4.6b}$$

$$A^{(3)}_\tau + V_3 \cdot A^{(3)}_\chi = \frac{\mathrm{i}\gamma}{D_\omega(\omega_3, \boldsymbol{\kappa}_3)} \bar{A}^{(1)} \bar{A}^{(2)}, \tag{4.6c}$$

where

$$V_r = -\frac{D_\kappa(\omega_r, \boldsymbol{\kappa}_r)}{D_\omega(\omega_r, \boldsymbol{\kappa}_r)} \tag{4.7}$$

is the group velocity of the rth wave. We remark that (4.5)–(4.7) are valid for contained triads (i.e. the energy of the waves remains finite) as well as explosive triads. We also emphasize that the required symmetry in (4.6) follows directly from the variational approach. Equations (4.6) possess conservation laws for wave action

$$D_\omega(\omega_1, \boldsymbol{\kappa}_1)\left(\frac{\partial}{\partial \tau} + V_1 \cdot \nabla\right)|A^{(1)}|^2 = D_\omega(\omega_2, \boldsymbol{\kappa}_2)\left(\frac{\partial}{\partial \tau} + V_2 \cdot \nabla\right)|A^{(2)}|^2$$

$$= D_\omega(\omega_3, \boldsymbol{\kappa}_3)\left(\frac{\partial}{\partial \tau} + V_3 \cdot \nabla\right)|A^{(3)}|^2, \tag{4.8}$$

and pseudoenergy

$$\omega_1 D_\omega(\omega_1, \boldsymbol{\kappa}_1)\left(\frac{\partial}{\partial \tau} + V_1 \cdot \nabla\right)|A^{(1)}|^2 + \omega_2 D_\omega(\omega_2, \boldsymbol{\kappa}_2)\left(\frac{\partial}{\partial \tau} + V_2 \cdot \nabla\right)|A^{(2)}|^2$$

$$+ \omega_3 D_\omega(\omega_3, \boldsymbol{\kappa}_3)\left(\frac{\partial}{\partial \tau} + V_3 \cdot \nabla\right)|A^{(3)}|^2 = 0 \tag{4.9a}$$

or

$$\left(\frac{\partial}{\partial \tau} + V_1 \cdot \nabla\right)E^{(1)} + \left(\frac{\partial}{\partial \tau} + V_2 \cdot \nabla\right)E^{(2)} + \left(\frac{\partial}{\partial \tau} + V_3 \cdot \nabla\right)E^{(3)} = 0, \tag{4.9b}$$

where $E^{(r)}$ is the pseudoenergy of the rth wave (see (3.10*b*)).

For $A^{(r)}$ purely a function of τ, (4.6) have been shown to admit explosive solutions (wherein the amplitudes of all triad members develop a singularity in a finite time) when the wave of largest frequency in absolute value has oppositely signed energy to the other two waves (cf. Craik 1985, §15 and (4.1), (4.8), (4.9)). For the partial differential equation (4.6), explosive solutions occur when the wave of highest frequency in absolute value has oppositely signed energy and travels with the middle group velocity (Kaup, Reiman & Bers 1979). These conditions for 'explosions' are Galilean invariant; hence, we argue that critical layers are necessary for the occurrence of explosive triads in continuously stratified shear flows. For while it is possible to choose a particular reference frame, defined so that $0 < \hat{U}_{min} < \hat{U}_{max}$, where $\hat{U}_{min/max}$

are the projections of the minimum and maximum speeds of the shear profile in the direction of the wavenumber vector $\boldsymbol{\kappa}$, in which waves of positive and negative pseudoenergy coexist without critical layers, a Galilean transformation to a reference frame for which $\hat{U}_{min} < 0 < \hat{U}_{max}$, ensures that all waves without critical layers have positive energy. Hence, for waves of oppositely signed pseudoenergy to persist in all reference frames (a necessary condition for the occurrence of explosive triads), at least one triad member must possess a critical layer. Then, in contrast to the layered-flow models, this critical-layer singularity must be resolved with either viscosity or nonlinearity. As viscosity may destabilize negative energy waves (cf. Ostrovskii *et al.* 1986), we anticipate that nonlinear critical layers will be relevant to this instability. It appears that a numerical study is necessary to determine whether explosive resonant triads exist in parametric domains for which the basic flow is linearly stable; however, this numerical study presents difficulties owing to the singularity at the critical layer and currently is under consideration.

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Appendix A. Layered models

To compare the present results (3.8*c*) and (4.4) to those obtained for the layered models referred to in §1, we introduce an interface at $z = z_0$ across which there may be discontinuities in $\rho_0(z)$ and $U_0(z)$. To determine the modifications to our theory caused by this interface, we replace this interface with a thin layer in which

$$\rho_0 = \rho_0(Z), \quad \hat{\omega} = \hat{\omega}(Z) = \omega - \boldsymbol{\kappa}\cdot \boldsymbol{U}_0(Z), \tag{A 1a, b}$$

$$Z \equiv (z-z_0)/\Delta \quad (-1 \leqslant Z \leqslant 1), \tag{A 1c}$$

and consider the limit $\Delta \to 0$. We expand the vertical modal structure of the wave (cf. (3.1*a*)) according to

$$\phi(z) = \Phi_0 + \Delta\psi(Z) + O(\Delta^2), \tag{A 2}$$

substitute into (3.4) and invoke (3.6*b*) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}Z}\left[\rho_0\hat{\omega}^2\frac{\mathrm{d}\psi}{\mathrm{d}Z}\right] \sim \frac{\kappa^2\Phi_0}{\beta}\frac{\mathrm{d}\rho_0}{\mathrm{d}Z}, \tag{A 3}$$

where, here and throughout this section, $\sim$ implies an approximation with an error of $O(\Delta)$. A first integral of (A 3) is

$$\frac{\rho_0\hat{\omega}^2}{\kappa^2}\frac{\mathrm{d}\psi}{\mathrm{d}Z} \sim \rho_0\Phi_0 + C - \tfrac{1}{2}(\rho_0^+ + \rho_0^-)\Phi_0, \tag{A 4a}$$

where
$$C = \frac{1}{2}\left[\frac{\rho_0(Z)\,\hat{\omega}(Z)^2}{\kappa^2}\frac{\mathrm{d}\psi}{\mathrm{d}Z}\right]^+, \quad [f(Z)]^+ \equiv f(1) + f(-1) \tag{A 4b}$$

and
$$\rho_0^{\pm} \equiv \rho_0(\pm 1). \tag{A 4c}$$

It follows that, as $\Delta \to 0$,

$$[\phi]_-^+ = 0, \quad \text{and} \quad \left[\frac{\rho_0\hat{\omega}^2}{\kappa^2}\phi'\right]_-^+ = [\rho_0]_-^+\phi/\beta, \tag{A 5a, b}$$

where $[\ ]_-^+$ denotes the jump across the interface $z = z_0$, in standard notation.

To determine the effect of an interface on the dispersion relationship, we substitute (A 2) into (3.8*c*) and invoke (A 1) to obtain

$$D(\omega,\boldsymbol{\kappa}) \sim D_{L1}+D_{L2}+D_I, \tag{A 6}$$

where
$$D_{L1} = \lim_{\Delta\to 0}\left\{\frac{1}{4\kappa^2}\int_{z_0+\Delta}^{d} \rho_0[\hat{\omega}^2\phi'^2+\kappa^2(\hat{\omega}^2-N^2)\phi^2]\,\mathrm{d}z\right\}, \tag{A 7a}$$

D_{L2} is given by (A 7*a*) with the range of integration replaced by $-d \leqslant z \leqslant z_0-\Delta$ and

$$D_I \sim \frac{1}{4}\int_{-1}^{1}\frac{\mathrm{d}\rho_0}{\mathrm{d}Z}\frac{\Phi_0^2}{\beta}\mathrm{d}Z = \tfrac{1}{4}\Phi_0^2/\beta[\rho_0^+-\rho_0^-]. \tag{A 7b}$$

As $\Delta\to 0$ we find that (3.8*c*) is replaced by

$$D(\omega,\boldsymbol{\kappa}) = \frac{1}{4\kappa^2}\int_{-d}^{d}\rho_0[\hat{\omega}^2\phi'^2+\kappa^2(\hat{\omega}^2-N^2)\phi^2]\,\mathrm{d}z+\frac{1}{4\beta}[\rho_0\,\phi^2]_-^+. \tag{A 8}$$

For more than one interface, the last term is replaced by a sum of terms, one for each interface. For example, for a three-layer fluid (with two interfaces) in which the density $\rho_0(z)$ and the velocity $U_0(z)$ are constant in each layer, (3.8*a*) is replaced by

$$\mathscr{L}_0 = D_+A_+\bar{A}_+ + D_-A_-\bar{A}_- + D_\pm(A_+\bar{A}_- + \bar{A}_+A_-), \tag{A 9}$$

where $A_\pm$ are the amplitudes of the waves on the upper/lower interface and $D_+ = 0$ is the dispersion relationship for waves on the upper interface if the lower interface is replaced by a rigid boundary and similarly for $D_- = 0$ (cf. Craik & Adams 1979). Invoking Hamilton's principle for $\mathscr{L}_0$ (A 7), we obtain

$$2D_+A_+ + D_\pm A_- = 0, \tag{A 10a}$$

and
$$2D_-A_- + D_\pm A_+ = 0. \tag{A 10b}$$

The dispersion relationship then is obtained by eliminating A_- from (A 10*a, b*) which yields the three-layer counterpart of (3.8*c*):

$$D(\omega,\boldsymbol{\kappa}) = -4D_+ + D_\pm^2/D_-. \tag{A 10c}$$

We next consider the effect of an interface on the nonlinear interaction coefficient (4.4). Substituting (A 2) into (4.4) and invoking (A 1), we obtain

$$\gamma \sim \lim_{\Delta\to 0}\{\gamma_{L1}+\gamma_{L2}\}+\gamma_I, \tag{A 11}$$

where $\gamma_{L1/L2}$ is given by (4.4) with the integration limits replaced by

$$(z_0+\Delta, d)/(-d, z_0-\Delta)$$

respectively and

$$8\gamma_I \sim \int_{-1}^{1}\rho_0\left\{\frac{\hat{\omega}_1^2}{\kappa_1^2}(\boldsymbol{\kappa}_2\cdot\boldsymbol{\kappa}_3)\,\psi_1'\left[\frac{\Phi_{20}\psi_3''}{\kappa_3^2}+\frac{\psi_2''\Phi_{30}}{\kappa_2^2}\right]+\frac{\hat{\omega}_2^2}{\kappa_2^2}(\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_3)\,\psi_2'\left[\frac{\Phi_{10}\psi_3''}{\kappa_3^2}+\frac{\psi_1''\Phi_{30}}{\kappa_1^2}\right]\right.$$
$$\left.+\frac{\hat{\omega}_3^2}{\kappa_3^2}(\boldsymbol{\kappa}_1\cdot\boldsymbol{\kappa}_2)\,\psi_3'\left[\frac{\Phi_{10}\psi_2''}{\kappa_2^2}+\frac{\psi_1''\Phi_{20}}{\kappa_1^2}\right]+\frac{1}{\beta}(\psi_1''\Phi_{20}\Phi_{30}+\Phi_{10}\psi_2''\Phi_3+\Phi_{10}\Phi_{20}\psi_2'')\right\}\mathrm{d}Z$$
$$-\frac{\rho_0}{\beta}(\psi_1'\Phi_{20}\Phi_{30}+\Phi_{10}\psi_2'\Phi_{30}+\Phi_{10}\Phi_{20}\psi_3')|_{-1}^{+1}, \tag{A 12}$$

where Φ_{r0} $(r = 1, 2, 3)$ is Φ_0, (A 2), for the rth wave. The generalization of this interaction coefficient for a three-layer flow is similar to that described above for the dispersion relationship (A 6) and is left to the reader.

For the particular (two-dimensional) two-layer flow considered in Tsutahara & Hashimoto (1986), for which

$$(\rho_0(z), U_0(z)) = (\rho_1, U_1\hat{x}) \quad (\Delta > z > \infty), \tag{A 13a}$$

and

$$(\rho_0(z), U_0(z)) = (\rho_2, U_2\hat{x}) \quad (-\infty < z < -\Delta), \tag{A 13b}$$

where $\hat{x}$ is a unit vector in the x-direction, we compute the dispersion relationship (A 6) and the nonlinear interaction coefficient (A 11). The vertical modal structure of the rth $(r = 1, 2, 3)$ wave superposed on the flow (A 13) is

$$\phi_r(z) = \Phi_{r0}\,\mathrm{e}^{-\kappa_r z} \quad (\Delta < z < \infty) \tag{A 14a}$$

and

$$\phi_r(z) = \Phi_{r0}\,\mathrm{e}^{\kappa_r z} \quad (-\infty < z < -\Delta). \tag{A 14b}$$

Substituting (A 13)–(A 14) into (A 7), we find

$$D_{L1} = \frac{\rho_1(\omega - kU_1)^2}{4\kappa}\Phi_0^2, \quad D_{L2} = \frac{\rho_2(\omega - kU_2)^2}{4\kappa}\Phi_0^2 \tag{A 15a, b}$$

and

$$D_I = \frac{\rho_1 - \rho_2}{4\beta}\Phi_0^2. \tag{A 15c}$$

Invoking (A 6), we recover Tsutahara & Hashimoto's (1) in the limit of zero surface tension.

We next compute (A 11) for a triad of waves that satisfy the resonance conditions (4.1) where we assume without loss of generality that $k_1 < 0$ and $k_2, k_3 > 0$. Substituting (A 13)–(A 14) into (A 11)–(A 12), we obtain

$$\lim_{\Delta \to 0}\{\gamma_{L1} + \gamma_{L2}\} = \tfrac{1}{4}[\rho_0(\hat{\omega}_2\kappa_3/\kappa_2 + \hat{\omega}_3\kappa_2/\kappa_3)]_-^+ \tag{A 16a}$$

and

$$\gamma_i = -\tfrac{1}{4}[\rho_0(\hat{\omega}_2\kappa_3/\kappa_2 + \hat{\omega}_3\kappa_2/\kappa_3) - \rho_0(\hat{\omega}_2\hat{\omega}_3)]_-^+ - \tfrac{1}{2}(\rho_1 - \rho_2)[(\phi_1\phi_2\phi_3)']^+. \tag{A 16b}$$

Invoking (A 11) and noting that the last term in (A 16*b*) is zero for the two-layer flow, we obtain

$$\gamma = \tfrac{1}{4}[\rho_0\hat{\omega}_2\hat{\omega}_3]_-^+ \tag{A 17}$$

which agrees with Tsutahara & Hasimoto's result (8) in the limit of zero surface tension.

Appendix B

We consider the Lagrangian as a function of the vector trial function $\boldsymbol{q}$ and its derivatives

$$L = L(\boldsymbol{q}, \boldsymbol{q}_t, \boldsymbol{q}_{\boldsymbol{x}}, \boldsymbol{x}, t), \tag{B 1}$$

where subscripts indicate partial differentiation and bold subscripts represent the gradient operator. In (2.13), $\boldsymbol{q} = (\xi, \eta, \zeta, q)$. We expand the trial function $\boldsymbol{q}$ in the small parameter ϵ according to

$$\boldsymbol{q} = \boldsymbol{q}_0 + \epsilon\boldsymbol{q}_1 + O(\epsilon^2), \tag{B 2}$$

which implies

$$\left.\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{q}_t} - \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\frac{\partial L}{\partial \boldsymbol{q}_{\boldsymbol{x}}}\right|_{\boldsymbol{q}=\boldsymbol{q}_0} = O(\epsilon) \tag{B 3}$$

We next expand the Lagrangian about $\boldsymbol{q} = \boldsymbol{q}_0$:

$$L(\boldsymbol{q}, \boldsymbol{q}_t, \boldsymbol{q}_x, \boldsymbol{x}, t) = L(\boldsymbol{q}_0, \boldsymbol{q}_{0t}, \boldsymbol{q}_{0x}, t) + \left.\frac{\partial L}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}_0} (\boldsymbol{q}-\boldsymbol{q}_0) + \left.\frac{\partial L}{\partial \boldsymbol{q}_t}\right|_{\boldsymbol{q}_0} (\boldsymbol{q}_t-\boldsymbol{q}_{0t}) + \left.\frac{\partial L}{\partial \boldsymbol{q}_x}\right|_{\boldsymbol{q}_0} (\boldsymbol{q}_x-\boldsymbol{q}_{0x}) + O((\boldsymbol{q}-\boldsymbol{q}_0)^2). \quad \text{(B 4)}$$

Defining the average Lagrangian $\langle L \rangle$

$$\langle L \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} L\,\mathrm{d}\theta, \quad \theta \equiv \boldsymbol{\kappa}\cdot\boldsymbol{x} - \omega t, \quad \text{(B 5)}$$

substituting (B 4) into (B 5), invoking

$$\frac{\mathrm{d}}{\mathrm{d}t} = -\omega\frac{\mathrm{d}}{\mathrm{d}\theta}, \quad \frac{\mathrm{d}}{\mathrm{d}x} = \kappa\frac{\mathrm{d}}{\mathrm{d}\theta} \quad \text{(B 6)}$$

and integrating the third and fourth terms on the right-hand side by parts, we obtain

$$2\pi\langle L \rangle = \int_0^{2\pi} L_0\,\mathrm{d}\theta + \int_0^{2\pi}\left[\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{q}_t} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial \boldsymbol{q}_x}\right]_{\boldsymbol{q}=\boldsymbol{q}_0} (\boldsymbol{q}-\boldsymbol{q}_0)\,\mathrm{d}\theta + O((\boldsymbol{q}-\boldsymbol{q}_0)^2). \quad \text{(B 7)}$$

Invoking (B 3) for the term in square brackets in (B 7), we find that the $O(\epsilon)$ error in approximating $\boldsymbol{q}$ by $\boldsymbol{q}_0$ in (B 5) leads to an error of $O(\epsilon^2)$ in $\langle L \rangle$.

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